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Generalization of Maeda's Theorem

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The theorem of S. Maeda concerning the characterization of finite measures on a quantum logic of all closed subspaces of a Hilbert space of dimension $\neq 2$ is generalized to the case of σ-finite measures with possible infinite values. The proof does not involve Gleason's result, but only the proposition on frame functions.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathscr{L}(H)$ be a quantum logic of all closed subspaces of a (not necessarily separable) Hilbert space H over the field $\mathbb{C}$ of real or complex numbers. A measure on $\mathscr{L}(H)$ is a function $m: \mathscr{L}(H) \to [0, \infty]$ such that (1) $m(0) = 0$; (2) m is σ-additive on all sequences of mutually orthogonal elements of $\mathscr{L}(H)$. Gleason's theorem (Gleason 1957) says that any finite measure m on a separable Hilbert space H, $\dim H \neq 2$, is in one-to-one correspondence with positive Hermitian operators T on H of finite trace via

$$m(M) = \operatorname{tr}(TM), \qquad M \in \mathscr{L}(H) \tag{1}$$

(we identify a subspace M with its orthoprojector P^M on it). Eilers and Horst (1975) and Drisch (1979) prove that the assumption of separability is superfluous when the Hilbert space is of dimension of nonmeasurable cardinality (for definition see below); consequently, any finite measure is already totally additive. Maeda (1980) (see also Kalmbach, 1983, p. 273) has given the characterization of all finite measures on a quantum logic $\mathscr{L}(H)$, $\dim H \neq 2$, showing that the following conditions are equivalent: (1) m is representable through a positive Hermitian operator T of finite trace via (1); (2) m has a support, i.e., there is an element $M \in \mathscr{L}(H)$ such that $m(N) = 0$ iff $N \perp M$; (3) m is totally additive on orthogonal elements of $\mathscr{L}(H)$. In proving that (3) implies (1), Maeda follows the proof in

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Gleason's paper, but he does not use the Gleason result. It is relatively easily verified that (1) implies (2), and (2) implies (3).

The situation with measures attaining infinite values is more complicated. These measures may appear in some descriptions of physical systems; for example, the dimension function is such a measure.

To formulate our results, we need the following notions. By $\mathrm{Tr}(H)$ we denote the class of all bounded operators T in H such that, for every orthonormal basis $\{x_a\colon a\in I\}$ of H, the series $\sum_{a\in I}(Tx_a, x_a)$ converges and is independent of the basis used; the expression $\mathrm{tr}\, T := \sum_{a\in I}(Tx_a, x_a)$ is called the trace of T.

A bilinear form is a function $t\colon D(t)\times D(t)\to C$ [$D(t)$ not necessarily dense or closed in H], called the domain of the definition of t, such that t is linear in both arguments, and $t(\alpha x, \beta y)=\alpha\bar{\beta}t(x, y)$, $x, y\in D(t)$, $\alpha, \beta\in C$. If $t(x, y)=\overline{t(y, x)}$ for all $x, y\in D(t)$, then t is said to be symmetric; if for a symmetric bilinear form t we have $t(x, x)\geq 0$ for all $x\in D(t)$, then t is said to be positive. Let $P\in\mathscr{L}(H)$ and let $P\subset D(t)$. Then by $t\circ P$ we mean a symmetric bilinear form defined by $t\circ P(x, y)=t(Px, Py)$, $x, y\in H$. If $t\circ P$ is induced by a trace operator T, that is, $t\circ P(x, y)=(Tx, y)$, $x, y\in H$, then we say $t\circ P\in\mathrm{Tr}(H)$ and we put $\mathrm{tr}\, t\circ P=\mathrm{tr}\, T$.

By $\bigoplus_{a\in I} M_a$ we mean the joint of mutually orthogonal elements $M_a\in\mathscr{L}(H)$, $a\in I$. If $0\neq x\in H$, then by P_x we denote the one-dimensional subspace of H spanned over x.

Let n be a cardinal. We say that a measure m is n-finite if there is a set I whose cardinal is n and a set of mutually orthogonal elements $\{M_a\colon a\in I\}\subset\mathscr{L}(H)$ such that $\bigoplus_{a\in I} M_a=H$ with $m(M_a)<\infty$, $a\in I$. If, in particular, $n=\aleph_0$ (i.e., the cardinal of the set of all integers), we say that m is σ-finite. For example, $m(M):=\dim M$, $M\in\mathscr{L}(H)$, is σ-finite iff H is separable.

Lugovaja and Sherstnev (1980) proved that for any σ-finite measure m on $\mathscr{L}(H)$, $m(H)=\infty$, of a separable Hilbert space H there exists a unique positive symmetric bilinear form t defined on a dense domain such that

$$m(P)=\begin{cases}\mathrm{tr}\, t\circ P & \text{iff } t\circ P\in\mathrm{Tr}(H)\\ \infty & \text{otherwise}\end{cases}\tag{2}$$

It is known that not any symmetric bilinear form determines via (2) a σ-finite measure. The necessary and sufficient condition for this is given by Lugovaja (1983).

2. MAEDA'S THEOREM

The crucial notion for our main goal is a frame function. Denote $S(H)=\{x\in H\colon \|x\|=1\}$. A function $f\colon S(H)\to[0,\infty]$ is a frame function if

(1) $f(\lambda x) = f(x)$ for all scalars λ with $|\lambda| = 1$; (2) there is a constant W (may be $+\infty$), called the weight of f, such that, for any orthonormal basis $\{x_a\colon a \in A\}$ of H, $\sum_{a\in A} f(x_a) = W$. A frame function f has a finiteness property if $\sum_{i\in I} f(x_i) < \infty$, for some orthogonal system of vectors $\{x_i\colon i \in I\} \subset H$, implies $f|S(G)$ is a frame function with a finite weight, where $G = \bigoplus_{i\in I} P_{x_i}$. It is clear that any frame function with a finite weight has the finiteness property. A frame function f is regular if there is a positive symmetric bilinear form t with $D(t) = \{x \in H\colon x \neq 0; f(x/\|x\|) < \infty\} \cup \{0\}$ such that $f(x) = t(x, x)$ for any $x \in S(H) \cap D(t)$. Let n be a cardinal. We say that a frame function f is n-finite if there exists an orthonormal basis, $\{x_a\colon a \in A\}$ such that $A = \bigcup_{i\in I} A_i$, where $A_i \cap A_j = \phi$ whenever $i \neq j$, $i, j \in I$, $\sum_{j\in A_i} f(x_j) < \infty$ for any $i \in I$, and the cardinal of I is n. In particular, if $n = \aleph_0$, then we say that f is σ-finite.

Lemma 1. Let f be a frame function with the finiteness property and with the infinite weight on $S(H)$ of a three-dimensional Hilbert space. If $f(x) + f(y) < \infty$ and $f(z) < \infty$, where $x \perp y$, then $z = \alpha x + \beta y$ for some scalars $\alpha, \beta \in C$.

Proof. If we put $m(0) = 0$, $m(P) = \sum_i m(P_{x_i})$, where $\{x_i\}$ is an orthonormal basis in P, then m is a measure on $\mathscr{L}(H)$ and the result follows from a lemma in Lugovaja and Sherstnev (1980). ■

Corollary 2. Let $3 \leq \dim H = n < \infty$ and let f be a frame function on $S(H)$ with the finiteness property and with infinite weight. If $f(x_1) + \cdots + f(x_{n-1}) < \infty$ and $f(z) < \infty$, where $x_i \perp x_j$, if $i \neq j$, then $z = \alpha_1 x_1 + \cdots + \alpha_n x_n$ for some scalars $\alpha_1, \ldots, \alpha_n \in C$.

Proof. Follows from Corollary 4.3 in Dvurečenskij (1985). ■

The cornerstone of the Gleason theorem is the assertion that any frame function with a finite weight on a three-dimensional real Hilbert space is regular. The proof is very nontrivial and many attempts at an elementary proof have been made (e.g., Gudder, 1982; Maljugin, 1982; Cooke *et al.*, 1985).

The following two results characterize frame functions with possible infinite values.

Theorem 3. Let $4 \leq \dim H < \infty$ and let f be a frame function on $S(H)$ with the finiteness property and with infinite weight. If there are three orthonormal vectors x, y, z such that $f(x) + f(y) + f(z) < \infty$, then f is regular.

Proof. Using Corollary 2, we see that if we put $M = \{x \in H\colon x \neq 0, f(x/\|x\|) < \infty\} \cup \{0\}$, then $M \in \mathscr{L}(H)$ and $\dim M \geq 3$. Using the known assertion on finite frame functions on finite-dimensional Hilbert space, we see that $f|S(M)$ is a regular frame function. ■

Theorem 4. Let H be a real or complex Hilbert space of dimension $\neq 2$ and let n be any cardinal. Then any n-finite frame function f with the finiteness property is regular.

Proof. If the weight of f is finite, then the assertion follows from the classical result of Gleason (1957).

Now let the weight of f be infinite. Define a map F on H via

$$F(x) = \begin{cases} 0 & x = 0 \\ f(x/\|x\|)\|x\|^2 & \text{for} \quad x \neq 0 \end{cases}$$

Put $D(F) = \{x \in H: F(x) < \infty\}$. We claim to show that $D(F)$ is a dense submanifold in H. Let x, $y \in D(F)$. Due to the n-finiteness of f, we have that there exist three orthonormal vectors x_1, x_2, x_3 and three scalars α_1, α_2, α_3 such that $f(x_1) + f(x_2) + f(x_3) < \infty$, $z := \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \perp x_1, x_2, x_3$, and $Px \neq 0 \neq Py$, where $P = \bigoplus_{i=1}^{3} P_{x_i}$. Due to Lemma 1, $f|S(M)$, where $M = P_z \vee P_x \vee P_y$, is a finite frame function; hence, $F(x+y) < \infty$. The density of $D(F)$ follows from the n-finiteness of f.

Now we define a positive symmetric bilinear form t. Since any two-dimensional subspace Q such that $f|S(Q)$ is a finite frame function, due to the n-finiteness and Theorem 3, may be embedded into some three-dimensional subspace N such that $f|S(N)$ is a finite frame function, $f|S(Q)$ is regular. Hence, there is a positive Hermitian operator $T_Q \in \mathrm{Tr}(H)$ such that $F(x, y) = (T_Q x, y)$ for all x, $y \in Q$.

Now let x, $y \in D(F)$. Define $t(x, y) = (T_Q x, y)$, where Q is some two-dimensional subspace of H containing x, y. It is easily verified that t is the well-defined symmetric positive bilinear form in question. Indeed, if x, $y \in Q_1$, Q_2, then

$$(T_{Q_1}x, x) = F(x) = (T_{Q_2}x, x) \quad ■$$

Our main goal is the following generalization of Maeda's theorem to measures with possible infinite values.

Theorem 5 (S. Maeda). Let $\mathscr{L}(H)$ be a quantum logic of a real or complex Hilbert space H of dimension $\neq 2$. Let n be a cardinal and let m be an n-finite measure. The following statements are equivalent:

1. There exists a unique positive bilinear form t with a dense domain $D(t)$ such that equation (2) holds.
2. m has a support.
3. m is totally additive.

Proof. Statement 1⇒2. Statement 1 implies that $D(t) = \{x \in H: m(P_x) < \infty\} \cup \{0\}$. Define $D_0 = \{x \in H: m(P_x) = 0\} \cup \{0\}$. We claim that D_0 is a closed submanifold in H. First, let $x, y \in D_0$. Since $D_0 \subset D(t)$, $x + y \in D(t)$. Check

$$t(x+y, x+y) = t(x, x) + t(x, y) + t(y, x) + t(y, y)$$

It is known that for any positive symmetric bilinear form t we have $|t(x, y)|^2 \leq t(x, x) \cdot t(y, y)$ for any $x, y \in D(t)$. Hence, $x + y \in D_0$.

Now we show that if $x_1, \ldots, x_n \in D_0$, then $m(M_n) = 0$, where $M_n = \bigvee_{i=1}^n P_{x_i}$. Without loss of generality we may assume $x_1, \ldots, x_n$ are linearly independent vectors. Applying the Gram–Schmidt orthogonalization process to $x_1, \ldots, x_n$, choose orthonormal vectors $y_i = \alpha_1^i x_1 + \cdots + \alpha_i^i x_i$, $i = 1, \ldots, n$. Then

$$\operatorname{tr} t \circ M_n = \sum_{i=1}^{n} t(M_n y_i, M_n y_i) = \textstyle\sum_{i=1}^n t(y_i, y_i) = 0$$

Due to statement 1, $m(M_n) = 0$.

To show that D_0 is a closed submanifold, consider a fundamental sequence $\{x_n\}_{n=1}^\infty \subset D_0$. Let $\|x - x_n\| \to 0$ when $n \to \infty$. Put $M_n = P_{x_1} \vee \cdots \vee P_{x_n}$; then $x \in M := \bigvee_{n=1}^\infty M_n$ and the continuity of m from below implies $m(M) = \lim_n m(M_n) = 0$, so that $x \in D_0$.

Now let $\{x_i: i \in I\}$ be any orthonormal basis in D_0 and $\{y_j: j \in J\}$ be any orthonormal basis in $D_0^\perp$. Check

$$\sum_{i \in I} t(D_0 x_i, D_0 x_i) + \sum_{j \in J} t(D_0 y_j, D_0 y_j) = 0$$

Consequently, $t \circ D_0 \in \operatorname{Tr}(H)$, and $m(D_0) = 0$. If we put $M = D_0^\perp$, then M is a unique support of m.

Statement 2⇒3. Let now $\{P_a: a \in A\}$ be an arbitrary system of mutually orthogonal elements of $\mathscr{L}(H)$ with the join P. If $m(\bigoplus_{a \in J} P_a) = \infty$ for some countable subset J of A, then $m(P) = \infty = \sum_{a \in A} m(P_a)$. Hence, suppose that $m(\bigoplus_{a \in J} P_a) < \infty$ for any countable subset J of A. Denote, for any $n \geq 1$, $A_n = \{a \in A: m(P_a) \geqslant 1/n\}$. Our assumption yields that any A_n is a finite subset of A. Put $A_0 = \bigcup_{n=1}^\infty A_n$. Then, for any $a \in A - A_0$, $m(P_a) = 0$; consequently, $P_a \perp M$, where M is a support of m. Therefore, $\bigoplus_{a \in A - A_0} P_a \perp M$ and $m(\bigoplus_{a \in A - A_0} P_a) = 0$. Since

$$m(P) = m\Big(\bigoplus_{a \in A - A_0} P_a\Big) + \sum_{a \in A_0} m(P_a)$$

we have $m(P) = \sum_{a \in A} m(P_a)$.

Statement 3⇒1. Define on $S(H)$ a function f via $f(x) = m(Px)$, $x \in S(H)$. Then f is an n-finite frame function with the finiteness property. Theorem 4 implies that there is a unique positive symmetric bilinear form

t with a dense domain $D(t)=\{x\in H\colon m(P_x)<\infty\}\cup\{0\}$ such that $f(x)=t(x,x)=m(P_x)$. Now we show that equation (2) holds. Let $m(P)<\infty$. If $\{x_i\}$ and $\{y_j\}$ are orthonormal bases in P and $P^{\perp}$, respectively, then the total additivity of m gives

$$m(P)=\sum_i m(P_{x_i})=\sum_i t(x_i,x_i)=\sum_i t(Px_i,Px_i)+\sum_j t(Py_j,Py_j)$$

which entails $t\circ P\in \mathrm{Tr}(H)$.

Conversely, if $t\circ P\in \mathrm{Tr}(H)$, then

$$\mathrm{tr}\; t\circ P=\sum_i t(x_i,x_i)=\sum_i m(P_{x_i})=m(P)$$

and the theorem is completely proved. ■

Remark. An immediate consequence of Theorem 5 is the Gleason theorem for σ-finite measures on a separable Hilbert-space quantum logic formulated by Lugovaja and Sherstnev (1980) [see (2)], since for a separable Hilbert space σ-additivity and total additivity coincide. Moreover, Theorem 5 says that in this case any σ-finite measure has a support.

Another application of Theorem 5 is Theorem 6 as follows. First we give the following definition. We say, according to Ulam (1930), that the cardinal I is nonmeasurable if there is no trivial positive finite measure ν on the power set a set A, whose cardinal is I, such that $\nu(\{a\})=0$ for any $a\in A$. In the opposite case I is called measurable cardinal. It is evident that any finite cardinal and $\aleph_0$ is nonmeasurable. It is known that if $J\le I$ and I is nonmeasurable, then so is J. If the continuum hypothesis holds (i.e., $\aleph_1=c$), then c (cardinal of reals) is nonmeasurable cardinal. Under the assumption of the generalized continuum hypothesis, the nonmeasurability of I implies the nonmeasurability of 2^I.

We say that the dimension of a Hilbert space H is a nonmeasurable cardinal if the cardinal of an orthonormal basis of H is nonmeasurable.

Let m be a cardinal. We say that a map $m:\mathscr{L}(H)\to[0,\infty]$ with $m(0)=0$ is m-additive if $m(\bigoplus_{t\in T}P_t)=\sum_{t\in T}m(P_t)$ whenever the cardinal of T is m.

Theorem 6. Let n and m be two cardinals such that $n\le m$, $\aleph_0\le m$. Then, for any n-finite m-additive measure m on a quantum logic $\mathscr{L}(H)$ of a Hilbert space H whose dimension is nonmeasurable cardinal $\neq 2$, each of the statements 1-3 of Theorem 5 holds.

Moreover, if M is a support of m, then $\dim M\le\max\{\aleph_0,n\}$.

Proof. We shall show that under our assumptions m has a support. This is true when m is a finite measure. Indeed, the results of Eilers and Horst (1975) and Drisch (1979) show that there is a unique positive Hermitian operator $T\in\mathrm{Tr}(H)$ such that $m(M)=\mathrm{tr}(TM)$, $M\in\mathscr{L}(H)$. Hence, according to Schatten (1970), $T=\sum_{a\in A}\lambda_a f_a\otimes\bar{f}_a$, where A is a countable index set, $f\times\bar{f}:x\mapsto(x,f)f$, for any $x\in H$, $\lambda_a>0$ for any $a\in A$. An easy calculation shows that $M:=\bigoplus_{a\in A}P_{f_a}$ is a support of m of dimension $\leq\aleph_0$.

Now let $m(H)=\infty$. The n-finiteness of m implies that there is a system of subspaces $\{H_i: i\in I\}$ such that $\bigoplus_{i\in I}H_i=H$, $m(H_i)<\infty$, for any $i\in I$, where the index set I has the cardinal n. Without loss of generality we may assume that dim $H_i\geq 3$. The first part of the present proof entails that, for any $i\in I$, $H_i=M_i\oplus N_i$, where M_i is a support of a finite measure $m_i:=m|\mathscr{L}(H_i)$, $i\in I$, with dim $M_i\leq\aleph_0$.

Let us put $H_\infty=\bigoplus_{i\in I}M_i$, $N_\infty=\bigoplus_{i\in I}N_i$; then dim $H_\infty=n$. Now we show that an n-finite n-additive measure $m_\infty:=m|\mathscr{L}(H_\infty)$ has a support of dimension $\leq n$. In fact, denote $D_0=\{x\in H_\infty: m_\infty(P_x)=0\}\cup\{0\}$. Theorem 4 entails the existence of a symmetric positive bilinear form t with a dense domain in H such that $m(P_x)=t(x,x)$ whenever $m(P_x)<\infty$. Therefore, as in the proof of the implication $1\Rightarrow 2$ from Theorem 5, x, $y\in D_0$ implies $x+y\in D_0$. Moreover, if $x_1,\ldots,x_n$ are linearly independent vectors belonging to D_0, then $m(M_n)=0$, where $M_n=\bigvee_{i=1}^{n}P_{x_i}$. Indeed, choosing orthonormal vectors $y_1,\ldots,y_n$ of form $y_i=\alpha_1^i x_1+\cdots+\alpha_i^i x_i$, $i=1,\ldots,n$, then

$$m(M_n)=\sum_{i=1}^{n}m(P_{y_i})=\sum_{i=1}^{n}t(y_i,y_i)=0$$

Now it is clear that D_0 is a closed submanifold in H_∞, and the n-additivity of m_∞ gives $m_\infty(D_0)=0$. Consequently, $M=H_\infty\wedge D_0^\perp$ is a support of m_∞, and dim $M\leq\max\{\aleph_0,n\}$.

Now we show that M is also a support of a measure m on $\mathscr{L}(H)$. Put $N=\{x\in H: m(P_x)=0\}\cup\{0\}$. Then as above $N\in\mathscr{L}(H)$. It is evident that $N_i\subset H$ for any $i\in I$, and $D_0\in N$. Then $M^\perp=D_0\oplus N_\infty\subset N$. We claim $N=M^\perp$. If not, then $x\in N\wedge M$. Simultaneously, $m(P_x)=0$ and $m(P_x)>0$, which gives a contradiction.

Finally, to prove the assertion of the theorem, it is necessary to apply Theorem 5. ∎

Proposition 7. Let m be an n-finite measure on a quantum logic $\mathscr{L}(H)$ of a Hilbert space of dimension $\neq 2$. If M is a support of m, then dim $M\leq\max\{\aleph_0,n\}$.

Proof. Theorem 5 implies that m is totally additive. Repeating the proof of Theorem 6, we obtain the assertion of the proposition. ∎

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